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Mixed convection with variable viscosity in a vertical annulus with uniform wall temperatures

Enzo Zanchini *

Universita' di Bologna, Dipartimento di Ingegneria Energetica, Nucleare e del Controllo Ambientale (DIENCA), Viale Risorgimento 2, 40136 Bologna, Italy

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Abstract

The steady and laminar mixed convection with a temperature-dependent viscosity in a vertical annular duct with uniform wall temperatures is studied analytically. The flow is considered as purely axial and the fluid density is assumed to be a linear function of temperature. Analytical expressions of the dimensionless velocity distribution, of the dimensionless pressure drop and of the Fanning friction factors are determined. The importance of choosing the mean fluid temperature as the reference temperature in the definition of the difference between the pressure and the hydrostatic pressure is pointed out. The results show that the combined effects of buoyancy forces and of a variable fluid viscosity on the cross-section-averaged Fanning friction factor may be important, and that negative values of this quantity may occur.

1. Introduction

The laminar mixed convection in vertical or inclined ducts has been widely studied in the literature. Indeed, this research area has several technical applications, such as heat exchangers, cooling systems for electronic devices, solar collectors. Interesting results of the researches on this topic are collected in Refs. [1,2].

With reference to mixed convection in vertical or inclined plane channels, the fully developed regime has been studied analytically by Aung and Worku [3], Cheng et al. [4], Hamadah and Wirtz [5], Barletta and Zanchini [6–8]; the flow stability has been investigated numerically by Chen and Chung [9,10]; the reversed flow has been studied experimentally by Gan et al. [11]. For mixed convection in vertical rectangular ducts, the fully developed region has been studied analytically by Barletta [12,13], while the entrance region has been investigated numerically by Cheng et al. [14,15]. The mixed convection in vertical or inclined circular tubes has been studied analytically in Refs. [16–19]; numerically in Refs. [20–23]; experimentally by Lavine et al. [24].

Some attention has been devoted also to mixed convection in vertical annular ducts. An analytical study of fully developed mixed convection in a vertical annulus with a uniform heat flux at each wall has been presented by Rokerya and Iqbal [25]. The authors have studied the effect of viscous dissipation on the Nusselt number with the following boundary conditions: outer wall heated and inner adiabatic, inner wall heated and outer adiabatic, both walls heated. Aung et al. [26] and Tson et al. [27] have presented numerical investigations of mixed convection in the entrance region of a vertical annular duct, with either inner wall heated and outer adiabatic [26], or outer wall heated and inner adiabatic [26], or both walls heated [26,27]. In particular, in Ref. [27] the authors have pointed out that, when both walls are heated, buoyancy forces may produce a strong increase of the cross-section-averaged Fanning friction factor in the entrance region of the duct, while this effect becomes vanishing in the fully developed region. Barletta [28] has studied analytically the fully developed mixed convection of a power-law fluid with constant properties in a vertical annular duct whose walls are kept at uniform but different temperatures.

* Tel.: +39 051 209 3295; fax: +39 051 209 3296.
E-mail address: enzo.zanchini@mail.ing.unibo.it

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Nomenclature

A, B	dimensionless coefficients
D	mean diameter, Eq. (10)
f_i, f_e, f_m	Fanning friction factors
f_{i0}, f_{e0}, f_{m0}	forced convection contributions to Fanning friction factors
f_{i1}, f_{e1}, f_{m1}	free convection contributions to Fanning friction factors
$f(r), g(r), h(r)$	dimensionless functions, Eq. (33)
$\mathbf{g}$	gravity acceleration
g	magnitude of the gravity acceleration
Gr	Grashof number, Eq. (9)
$l(r), m(r)$	dimensionless functions, Eq. (33)
p	pressure
P	difference between the pressure and the hydrostatic pressure, Eq. (5)
R	radial coordinate
R_1	radius of the inner wall
R_2	radius of the outer wall
r	dimensionless radial coordinate, Eq. (9)
Re	Reynolds number, Eq. (9)
T	temperature
T_0	mean fluid temperature, Eq. (3)
T_1	temperature of the inner wall
T_2	temperature of the outer wall
T_r	reference temperature
$\mathbf{U}$	velocity
U	axial velocity
u	dimensionless axial velocity, Eq. (9)
u_0	forced convection contribution to u
u_1	free convection contribution to u
u_B, u_{mB}	dimensionless velocities defined in Ref. [28]
u_{0B}, u_{m0B}	forced convection contributions to u_B, u_{mB}
u_{1B}, u_{m1B}	free convection contributions to u_B, u_{mB}
$\mathbf{x}$	unit vector along the duct axis
X	axial coordinate
Greek symbols	
γ	dimensionless parameter, Eq. (9)
ζ	dimensionless parameter, Eq. (22)
θ	dimensionless temperature, Eq. (9)
Λ	dimensionless parameter defined in Ref. [28]
λ	dimensionless pressure drop, Eq. (9)
λ_0	forced convection contribution to λ
λ_1	free convection contribution to λ
μ	dynamic viscosity
μ_0	dynamic viscosity for $T = T_0$
μ_r	dynamic viscosity for $T = T_r$
ξ	dimensionless parameter, Eq. (22)
ρ	mass density
ρ_0	mass density for $T = T_0$
Φ	dimensionless viscosity, Eq. (9)
ω	dimensionless parameter, Eq. (9)

The motivation for the present investigation is given by the results found in Refs. [8,28]. In Ref. [8], the authors have analyzed the combined action of buoyancy forces and of variable viscosity on the velocity distribution and on the Fanning friction factors, for a vertical or inclined plane channel with uniform wall temperatures. The results show that, while for an ideal fluid with constant viscosity the cross-section-averaged Fanning friction factor is independent of buoyancy forces, for a fluid with temperature-dependent viscosity the viscous pressure drop depends on the ratio between the Grashof number and the Reynolds number. In Ref. [28], the author has shown that for a Newtonian fluid which flows in a vertical annular duct with uniform wall temperatures, even in the scheme of constant viscosity the cross-section-averaged Fanning friction factor is sharply influenced by buoyancy forces.

In the present paper, the combined effect of buoyancy forces and of a temperature-dependent viscosity on the velocity distribution and on the Fanning friction factors is studied analytically, with reference to the fully developed laminar mixed convection in a vertical annular duct with uniform wall temperatures. The results show that, in most conditions, the variable viscosity enhances the dependence of the viscous pressure drop on buoyancy. In some cases, negative values of the cross-section-averaged Fanning friction factor occur. In fact, through the mechanism of flow reversal, the heat flow from the warm to the cool wall can act as a thermal pump, i.e., it can produce an increase of the difference between the pressure and the hydrostatic pressure along the flow direction.

2. Analytical solution

Let us consider the vertical annular duct represented in Fig. 1 and refer to a cylindrical coordinate system (X, R, Θ), such that X is the duct axis. Let us refer to steady, parallel and fully developed laminar flow; let us assume that the internal wall, with radius R_1, is kept at a uniform and constant temperature T_1, while the external wall, with radius R_2, is kept at a uniform and constant temperature T_2. The kind of flow considered implies that only the X-component U of the velocity vector U is non-zero; moreover, on account of the thermal boundary condition, the fluid temperature T is independent of X. Let us assume that the gravity acceleration, with a magnitude g, is opposite to the unit vector $\mathbf{x}$ in the positive X direction, i.e., that

$$\mathbf{g} = -g\mathbf{x}, \tag{1}$$

as is shown in Fig. 1. Finally, let us assume that the mass density of the fluid is a linear function of temperature, according to the equation of state

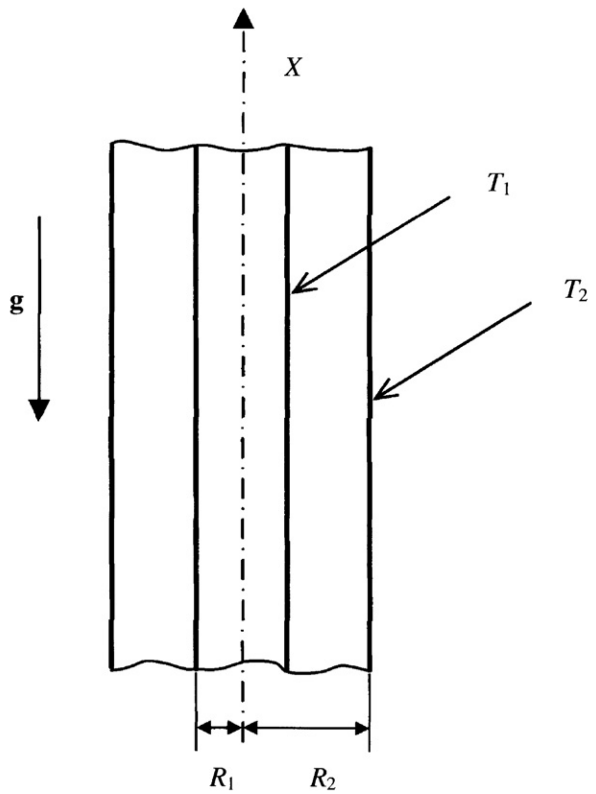

Fig. 1. Drawing of the duct.

$$\rho = \rho_0[1 - \beta(T - T_0)]. \tag{2}$$

In Eq. (2), T_0 is the reference fluid temperature, ρ is the fluid mass density, ρ_0 is mass density at temperature T_0 and β is the thermal expansion coefficient. As is suggested in Ref. [6], we will choose as reference temperature the mean fluid temperature in any cross-section of the duct, i.e.,

$$T_0 = \frac{2}{R_2^2 - R_1^2}\int_{R_1}^{R_2} TR\,\mathrm{d}R. \tag{3}$$

Since $\mathbf{U} = (U, 0, 0)$, the mass balance equation can be written as

$$\frac{\partial}{\partial X}(\varrho U) = 0. \tag{4}$$

Since ϱ is independent of X, Eq. (4) implies that $\partial U/\partial X = 0$. Thus, on account of the axial symmetry of the problem, U depends only on R. Let us define the difference P between the pressure p and the hydrostatic pressure as

$$P = p + \rho_0 g X. \tag{5}$$

The momentum balance equation in the radial direction yields $\partial P/\partial R = 0$, and ensures that P depends only on X. The momentum balance equation in the axial direction and the energy balance equation can be written as

$$\varrho_0 g\beta(T - T_0) - \frac{\mathrm{d}P}{\mathrm{d}X} + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(\mu R\frac{\mathrm{d}U}{\mathrm{d}R}\right) = 0, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}T}{\mathrm{d}R}\right) = 0, \tag{7}$$

where μ is the fluid viscosity. Eq. (7) shows that, since the flow is purely axial and $\partial U/\partial X = 0$, the temperature distribution is the same as in the case of heat conduction, i.e., is independent of the velocity field. By differentiating Eq. (6) with respect to X, one obtains $\mathrm{d}^2P/\mathrm{d}X^2 = 0$ and proves that $\mathrm{d}P/\mathrm{d}X$ is a constant. The boundary conditions for U and T are

$$U(R_1) = U(R_2) = 0, \quad T(R_1) = T_1, \quad T(R_2) = T_2. \tag{8}$$

Let us introduce the following dimensionless variables:

$$r = \frac{R}{R_2}, \quad \gamma = \frac{R_1}{R_2}, \quad u = \frac{U}{U_0}, \quad \theta = \frac{T - T_0}{T_1 - T_2},$$
$$\omega = \frac{T_1 - T_0}{T_1 - T_2}, \quad \lambda = -\frac{D^2}{\mu_0 U_0}\frac{\mathrm{d}P}{\mathrm{d}X}, \quad Re = \frac{\rho_0 U_0 D}{\mu_0},$$
$$Gr = \frac{g\beta\varrho_0^2(T_1 - T_2)D^3}{\mu_0^2}, \quad \Phi = \frac{\mu}{\mu_0}. \tag{9}$$

In Eq. (9), μ_0 is the fluid viscosity at $T = T_0$; D and U_0 are the mean diameter of the duct and the mean velocity of the fluid, which can be expressed as

$$D = 2R_2(1 - \gamma), \quad U_0 = \frac{2}{R_2^2 - R_1^2}\int_{R_1}^{R_2} UR\,\mathrm{d}R. \tag{10}$$

If U_0 is positive, we will say that the flow is upward; if it is negative, we will say that the flow is downward. From Eq. (9) one obtains also

$$\frac{Gr}{Re} = \frac{g\beta\varrho_0(T_1 - T_2)D^2}{\mu_0 U_0}. \tag{11}$$

Clearly, Gr has the sign of $T_1 - T_2$, while Gr/Re has the sign of $(T_1 - T_2)/U_0$. By employing Eq. (9), one can write Eqs. (6) and (7) in the dimensionless form

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -\frac{1}{4(1-\gamma)^2}\frac{Gr}{Re}\theta - \frac{\lambda}{4(1-\gamma)^2}, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 0. \tag{13}$$

On account of Eqs. (8) and (9), the dimensionless boundary conditions are

$$u(\gamma) = u(1) = 0, \quad \theta(\gamma) = \omega, \quad \theta(1) = \omega - 1. \tag{14}$$

The integration of Eq. (13), with the boundary condition (14), yields

$$\theta(r) = \frac{\ln(r)}{\ln(\gamma)} + \omega - 1. \tag{15}$$

From Eqs. (3) and (9), one obtains the following constraint on θ:

$$\int_\gamma^1 \theta(r)r\,\mathrm{d}r = 0. \tag{16}$$

Eqs. (15) and (16) yield

$$\omega = \frac{1}{1-\gamma^2} + \frac{1}{2\ln(\gamma)}. \tag{17}$$

From Eqs. (15) and (17) one obtains

$$\theta(r) = \frac{\ln(r)}{\ln(\gamma)} + \frac{\gamma^2}{1-\gamma^2} + \frac{1}{2\ln(\gamma)}. \tag{18}$$

In order to determine the dimensionless velocity field, one must find an expression of Φ as a function of r. As in Ref. [8], we will assume that the dependence of μ on T is described by the relation

$$\ln\frac{\mu}{\mu_r} = A(T - T_r) + B(T - T_r)^2, \tag{19}$$

where the reference temperature T_r will be chosen as $T_r = (T_1 + T_2)/2$. Eq. (19) is an improvement of the exponential relation between viscosity and temperature usually considered in the literature, such as in Refs. [29,30]. As is shown in Ref. [8]. Eq. (19) provides a very accurate description of the temperature dependence of viscosity for several liquids, in temperature ranges of 40 K. For a given fluid, the interpolation coefficients A and B depend only on the extreme temperatures T_2 and T_1 of the interpolation range. Eqs. (9) and (19) yield

$$\Phi = \frac{\mu}{\mu_r}\frac{\mu_r}{\mu_0} = \frac{\exp[A(T - T_r) + B(T - T_r)^2]}{\exp[A(T_0 - T_r) + B(T_0 - T_r)^2]}, \tag{20}$$

i.e.,

$$\Phi = \exp\{[A + 2B(T_0 - T_r)](T_1 - T_2)\theta + B(T_1 - T_2)^2\theta^2\}. \tag{21}$$

By introducing the dimensionless coefficients

$$\xi = [A + 2B(T_0 - T_r)](T_1 - T_2), \quad \zeta = B(T_1 - T_2)^2, \tag{22}$$

one can write Eq. (21) in the dimensionless form

$$\Phi = \exp[\xi\theta + \zeta\theta^2]. \tag{23}$$

Eqs. (18) and (23) yield

$$\Phi(r) = \exp\left\{\frac{\zeta}{[\ln(\gamma)]^2}[\ln(r)]^2 + \left[\frac{\xi}{\ln(\gamma)} + \frac{2\zeta}{\ln(\gamma)}\left[\frac{\gamma^2}{1-\gamma^2} + \frac{1}{2\ln(\gamma)}\right]\ln(r)\right]\right\} \times \exp\left\{\xi\left[\frac{\gamma^2}{1-\gamma^2} + \frac{1}{2\ln(\gamma)}\right] + \zeta\left[\frac{\gamma^2}{1-\gamma^2} + \frac{1}{2\ln(\gamma)}\right]^2\right\}. \tag{24}$$

By integrating Eq. (12) with $\Phi(r)$ given by Eq. (24) and with the boundary condition (14), one obtains

$$u(r) = \frac{\lambda}{8(1-\gamma)^2}\left\{\frac{\int_\gamma^1 [\Phi(r)]^{-1} r\,dr}{\int_\gamma^1 [\Phi(r)r]^{-1} dr}\int_\gamma^r [\Phi(r')r']^{-1} dr' - \int_\gamma^r [\Phi(r')]^{-1} r'\,dr'\right\} + \frac{Gr}{Re}\frac{1}{4(1-\gamma)^2}\left\{\frac{\int_\gamma^1 [\Phi(r)r]^{-1} dr \int_\gamma^r \theta(r')r'\,dr'}{\int_\gamma^1 [\Phi(r)r]^{-1} dr} \times \int_\gamma^r [\Phi(r')r']^{-1} dr' - \int_\gamma^r [\Phi(r')r']^{-1} dr' \int_\gamma^{r'} \theta(z) z\,dz\right\}, \tag{25}$$

where r' and z are integration variables. On account, of Eqs. (10) and (9), the dimensionless velocity $u(r)$ must fulfill the constraint

$$\int_\gamma^1 u(r) r\,dr = \frac{1-\gamma^2}{2}. \tag{26}$$

By applying the constraint (26) to Eq. (25), one obtains the expression of λ. By substituting the latter in Eq. (25), one finds the expression of u. After suitable simplifications, which include the use of integration by parts, one obtains

$$\lambda = \lambda_0 + \frac{Gr}{Re}\lambda_1, \tag{27}$$

$$\lambda_0 = \frac{8(1-\gamma^2)(1-\gamma)^2 g(1)}{g(1)l(1) - [f(1)]^2}, \tag{28}$$

$$\lambda_1 = \frac{2[f(1)h(1) - g(1)m(1)]}{g(1)l(1) - [f(1)]^2}, \tag{29}$$

$$u = u_0 + \frac{Gr}{Re}u_1, \tag{30}$$

$$u_0 = \frac{\lambda_0[f(1)g(r) - g(1)f(r)]}{8(1-\gamma)^2 g(1)}, \tag{31}$$

$$u_1 = \frac{\lambda_1[f(1)g(r) - g(1)f(r)]}{8(1-\gamma)^2 g(1)} + \frac{h(1)g(r) - g(1)h(r)}{4(1-\gamma)^2 g(1)}, \tag{32}$$

where

$$f(r) = \int_\gamma^r \frac{z\,dz}{\Phi(z)}, \quad g(r) = \int_\gamma^r \frac{dz}{\Phi(z)z},$$
$$h(r) = \int_\gamma^r \frac{dr'}{\Phi(r')r'}\int_\gamma^{r'} \theta(z) z\,dz, \quad l(r) = \int_\gamma^r \frac{z^3\,dz}{\Phi(z)}, \tag{33}$$
$$m(r) = \int_\gamma^r \frac{r'\,dr'}{\Phi(r')}\int_\gamma^{r'} \theta(z) z\,dz.$$

In the following, λ will be called dimensionless pressure drop. Eqs. (27)–(29) and (33) show that the dimensionless pressure drop can be written as the sum of two terms. The first term, λ_0, is the dimensionless pressure drop which occurs in forced convection, i.e., when the ratio Gr/Re vanishes. The second term is proportional to Gr/Re through the coefficient λ_1, which represents the influence of buoyancy forces on the dimensionless pressure drop. Since λ_0 and λ_1 depend on γ, ξ and ζ, then λ depends on γ, ξ, ζ and Gr/Re. The parameters ξ and ζ describe the effect of the temperature dependence of the dynamic viscosity. For a given fluid, these coefficients are determined by T_1, T_2 and γ. Similarly, Eqs. (28)–(32) show that u can be written as the sum of two terms. The first term, u_0, is the dimensionless velocity which occurs in forced convection, while the second term is proportional to Gr/Re through the coefficient u_1, which represents the contribution of buoyancy forces to the dimensionless velocity distribution.

For an annular duct, one can define the Fanning friction factor at the inner wall, f_i, and the Fanning friction factor at the outer wall, f_e, as follows:

$$f_i = \frac{2\tau_w(R_1)}{\rho_0 U_0^2}, \tag{34}$$

$$f_e = -\frac{2\tau_w(R_2)}{\rho_0 U_0^2}, \tag{35}$$

where the shear stress at the inner wall $\tau_w(R_1)$ and the shear stress at the outer wall $\tau_w(R_2)$ can be expressed as

$$\tau_w(R_1) = \left(\mu\frac{dU}{dR}\right)\bigg|_{R=R_1}, \quad \tau_w(R_2) = \left(\mu\frac{dU}{dR}\right)\bigg|_{R=R_2}. \tag{36}$$

From Eqs. (34)–(36), (9) and (10) one obtains

$$f_i Re = 4(1-\gamma)\Phi(\gamma)\left(\frac{du}{dr}\right)\bigg|_{r=\gamma}, \tag{37}$$

$$f_e Re = -4(1-\gamma)\Phi(1)\left(\frac{du}{dr}\right)\bigg|_{r=1}. \tag{38}$$

If one multiplies Eq. (37) by γ and adds the result to Eq. (38), one obtains

$$\Phi(\gamma)\gamma\left(\frac{du}{dr}\right)\bigg|_{r=\gamma} - \Phi(1)\left(\frac{du}{dr}\right)\bigg|_{r=1} = \frac{(\gamma f_i + f_e)Re}{4(1-\gamma)}. \tag{39}$$

By integrating Eq. (12) over the duct cross-section and by employing Eq. (16), one obtains

$$\Phi(\gamma)\gamma\left(\frac{du}{dr}\right)\bigg|_{r=\gamma} - \Phi(1)\left(\frac{du}{dr}\right)\bigg|_{r=1} = \lambda\frac{(1-\gamma^2)}{8(1-\gamma)^2}. \tag{40}$$

Clearly, Eqs. (39) and (40) yield

$$\lambda = \frac{2(\gamma f_i + f_e)Re}{1+\gamma}. \tag{41}$$

If, as usual, one defines the cross-section-averaged Fanning friction factor f_m as

$$f_m = \frac{f_i\gamma + f_e}{1+\gamma}, \tag{42}$$

from Eqs. (41) and (42) one obtains

$$\lambda = 2 f_m Re. \tag{43}$$

Note that the proportionality between λ and $f_m Re$, stated by Eq. (43), holds only if the mean temperature defined by Eq. (3) is chosen as the reference fluid temperature, i.e., if Eq. (16) holds. If one chooses a different reference temperature, one determines a dimensionless pressure drop which is not proportional to $f_m Re$ and thus has a poor physical meaning. This argument supports the choice of the reference fluid temperature recommended in Ref. [6].

By employing Eqs. (30)–(32), (37) and (38) one finds the following expressions of $f_i Re$ and $f_e Re$:

$$f_i Re = f_{i0}Re + \frac{Gr}{Re} f_{i1}Re, \tag{44}$$

$$f_e Re = f_{e0}Re + \frac{Gr}{Re} f_{e1}Re, \tag{45}$$

where

$$f_{i0}Re = \frac{\lambda_0[f(1) - \gamma^2 g(1)]}{2\gamma(1-\gamma)g(1)}, \tag{46}$$

$$f_{i1}Re = \frac{\lambda_1[f(1) - \gamma^2 g(1)] + 2h(1)}{2\gamma(1-\gamma)g(1)}, \tag{47}$$

$$f_{e0}Re = -\frac{\lambda_0[f(1) - g(1)]}{2(1-\gamma)g(1)}, \tag{48}$$

$$f_{e1}Re = -\frac{\lambda_1[f(1) - g(1)] + 2h(1)}{2(1-\gamma)g(1)}. \tag{49}$$

3. Limiting case of constant viscosity

In order to check the validity of the solution presented above, it is interesting to consider the limiting case of constant viscosity, which has been studied in Ref. [28]. In the limit of constant viscosity, i.e., $\Phi = 1$, from Eq. (33) one obtains

$$\begin{aligned} f(r) &= \frac{r^2-\gamma^2}{2}, \quad g(r) = \ln\frac{r}{\gamma}, \\ h(r) &= \frac{(\gamma^2-1)(\gamma^2-r^2+2r^2\ln r) - 2\gamma^2\ln\gamma(r^2-1-2\ln r+2\ln\gamma)}{8(\gamma^2-1)\ln\gamma}, \\ l(1) &= \frac{1-\gamma^4}{4}, \quad m(1) = \frac{\gamma^4-1-4\gamma^2\ln\gamma}{32\ln\gamma}. \end{aligned} \tag{50}$$

Eqs. (28), (29), (31), (32), (46)–(50) yield

$$\lambda_0 = \frac{32(\gamma-1)^2\ln\gamma}{1-\gamma^2-(1+\gamma^2)\ln\gamma}, \tag{51}$$

$$\lambda_1 = \frac{\ln\gamma(-1+\gamma^4+4\gamma^2\ln\gamma) - 2(\gamma^2-1)^2}{4(\gamma^2-1)\ln\gamma[1-\gamma^2+(1+\gamma^2)\ln\gamma]}, \tag{52}$$

$$u_0 = \frac{2[(\gamma^2-1)\ln r - (r^2-1)\ln\gamma]}{1-\gamma^2+(1+\gamma^2)\ln\gamma}, \tag{53}$$

$$\begin{aligned} u_1 = &\frac{[(\gamma^2-1)\ln r - (r^2-1)\ln\gamma][\ln\gamma(\gamma^4-1+4\gamma^2\ln\gamma) - 2(\gamma^2-1)^2]}{64(\gamma-1)^3(1+\gamma)(\ln\gamma)^2[1-\gamma^2+(1+\gamma^2)\ln\gamma]} \\ &+ \frac{(\gamma^2-1)\ln r(1-\gamma^2-2r^2\ln\gamma) + (r^2-1)\ln\gamma(\gamma^2-1+2\gamma^2\ln\gamma)}{32(\gamma-1)^3(1+\gamma)(\ln\gamma)^2}. \end{aligned} \tag{54}$$

$$f_{i0}Re = \frac{8(\gamma-1)(1-\gamma^2+2\gamma^2\ln\gamma)}{\gamma[1-\gamma^2+(1+\gamma^2)\ln\gamma]}, \tag{55}$$

$$f_{i1}Re = \frac{1-5\gamma^2+7\gamma^4-3\gamma^6+2\gamma^2(-3+2\gamma^2+\gamma^4-4\ln\gamma)\ln\gamma}{16(\gamma-1)^2\gamma(1+\gamma)\ln\gamma[1-\gamma^2+(1+\gamma^2)\ln(\gamma)]}, \tag{56}$$

$$f_{e0}Re = \frac{8(\gamma-1)(\gamma^2-1-2\ln\gamma)}{1-\gamma^2+(1+\gamma^2)\ln\gamma}, \tag{57}$$

$$f_{e1}Re = \frac{(3-\gamma^2)(\gamma^2-1)^2 + (2+4\gamma^2-6\gamma^4)\ln\gamma + 8\gamma^4(\ln\gamma)^2}{16(\gamma-1)^2(1+\gamma)\ln\gamma[1-\gamma^2+(1+\gamma^2)\ln\gamma]}. \tag{58}$$

The comparison of Eqs. (51)–(58) with the results presented in Ref. [28] is not immediate, because the reference velocity considered there is not the average velocity of the fluid in any cross-section, but a quantity proportional to the viscous pressure drop per unit length. In Ref. [28], a dimensionless velocity that will be denoted by u_B and a mean dimensionless velocity that will be denoted by u_{mB} are defined, and expressions of these quantities are reported, in the form

$$u_B = u_{0B} + \Lambda u_{1B}, \tag{59}$$

$$u_{mB} = u_{m0B} + \Lambda u_{m1B}, \tag{60}$$

where Λ is a dimensionless parameter related to the ratio Gr/Re defined here by the equation

$$\frac{Gr}{Re} = -\frac{\Lambda}{u_{mB}}. \tag{61}$$

It can be easily proved that the following relations hold

$$u_0 = \frac{u_{0B}}{u_{m0B}}, \quad u_1 = u_{0B}\frac{u_{m1B}}{u_{m0B}}, \tag{62}$$

$$\lambda_0 = \frac{1}{2u_{m0B}}, \quad \lambda_1 = \frac{u_{m1B}}{2u_{m0B}}. \tag{63}$$

By substituting in Eqs. (62) and (63) the expressions of u_{0B}, u_{1B}, u_{m0B} and u_{m1B} reported in Ref. [28], one obtains Eqs. (51)–(54). In a similar way, one can deduce Eqs. (55)–(58) from the results presented in Ref. [28]. The agreement of the results presented here with those reported in Ref. [28], in the special case of a Newtonian fluid with constant viscosity, provides a cross validation of these works.

4. Results

In the cases considered in this paper, the fluid flow is purely axial while the heat now is purely radial, so that no temperature change occurs in the flow direction and the temperature distribution is the same as in the case of pure conduction. Thus, the heat transfer problem has a trivial solution and will not be discussed. Moreover, analytical expressions of the velocity distribution, of the dimensionless pressure drop and of the Fanning friction factors have been provided. From these expressions, values of λ, f_iRe, f_eRe and plots of the velocity distribution can be easily obtained. Therefore, our discussion will illustrate only the most important results; in particular, our attention will be focused on the effect of buoyancy forces and of the variable viscosity on the dimensionless pressure drop λ, which is proportional to f_mRe, as is shown by Eq. (43).

Four different physical conditions will be considered: internal wall heated and upward flow; internal wall heated and downward flow; external wall heated and upward flow; external wall heated and downward flow. As for the sign of the coefficients ξ and ζ defined in Eq. (22), we will assume that ξ has a sign opposite to that of $T_1 - T_2$ and ζ is positive. These assumptions hold for most liquids, in temperature ranges not too far from room temperatures. The parameter Gr/Re has the sign of $(T_1 - T_2)/U_0$, where U_0 is positive for upward flow and negative for downward flow. Therefore, the physical conditions described above correspond to the following signs of the dimensionless parameters ξ and Gr/Re.

If the internal wall is heated, i.e., $T_1 > T_2, \xi$ is negative, while Gr/Re is positive for upward flow and negative for downward flow. On the other hand, if the external wall is heated, i.e., $T_1 < T_2, \xi$ is positive, while Gr/Re is negative for upward flow and positive for downward flow.

By employing Eqs. (28) and (33), the values of λ_0 as a function of γ and ξ have been calculated for $\zeta = 0.0$, $\zeta = 0.2$ and $\zeta = 0.4$. The results are collected in Tables 1–3, respectively. The analysis of these tables shows that λ_0 is an increasing function of ξ for small values of γ. With reference to the values of ζ considered in the tables, this circumstance occurs for $\gamma \leqslant 0.4$. On the other hand, λ_0 is not a monotonic function of ξ for higher values of γ. In the limit $\gamma \to 1$, the annular duct tends to a parallel plane channel. In this limit, the function $\lambda_0(\xi)$ is symmetric with respect to $\xi = 0$, as expected.

Table 1
Values of λ_0 with $\zeta = 0.0$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	41.11	41.29	41.67	42.28	43.13	44.25	45.61	47.20	48.99
0.100	42.57	42.77	43.17	43.80	44.69	45.83	47.23	48.86	50.69
0.150	43.53	43.73	44.11	44.72	45.58	46.69	48.06	49.64	51.41
0.200	44.28	44.45	44.80	45.37	46.18	47.23	48.53	50.04	51.71
0.250	44.90	45.04	45.35	45.86	46.60	47.59	48.81	50.23	51.81
0.300	45.43	45.54	45.79	46.24	46.92	47.84	48.98	50.31	51.78
0.350	45.90	45.96	46.16	46.56	47.17	48.01	49.06	50.30	51.67
0.400	46.32	46.33	46.48	46.81	47.36	48.12	49.10	50.25	51.53
0.500	47.04	46.96	47.00	47.21	47.63	48.25	49.08	50.07	51.17
0.600	47.65	47.47	47.41	47.50	47.79	48.29	48.98	49.83	50.78
0.700	48.16	47.90	47.73	47.72	47.90	48.28	48.85	49.58	50.39
0.800	48.61	48.25	47.99	47.88	47.96	48.24	48.70	49.32	50.02
0.900	49.00	48.56	48.21	48.01	47.99	48.17	48.54	49.06	49.67
0.999	49.34	48.81	48.38	48.10	48.00	48.10	48.38	48.82	49.34

Table 2
Values of λ_0 with $\xi = 0.2$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	41.68	41.91	42.35	43.02	43.94	45.11	46.51	48.13	49.94
0.100	43.30	43.56	44.02	44.72	45.66	46.86	48.30	49.96	51.79
0.150	44.38	44.64	45.08	45.75	46.66	47.83	49.23	50.84	52.61
0.200	45.22	45.45	45.86	46.48	47.34	48.44	49.77	51.30	52.98
0.250	45.91	46.11	46.47	47.04	47.83	48.86	50.11	51.54	53.11
0.300	46.51	46.67	46.97	47.48	48.20	49.15	50.31	51.65	53.11
0.350	47.02	47.14	47.39	47.83	48.48	49.35	50.43	51.67	53.03
0.400	47.48	47.55	47.75	48.12	48.70	49.50	50.49	51.64	52.90
0.500	48.27	48.24	48.33	48.57	49.02	49.66	50.50	51.48	52.56
0.600	48.93	48.80	48.77	48.90	49.21	49.73	50.42	51.26	52.18
0.700	49.48	49.26	49.13	49.14	49.34	49.73	50.29	51.00	51.79
0.800	49.95	49.64	49.41	49.32	49.41	49.69	50.15	50.74	51.41
0.900	50.36	49.96	49.63	49.45	49.45	49.63	49.99	50.48	51.05
0.999	50.71	50.22	49.82	49.55	49.46	49.55	49.82	50.23	50.72

Table 3
Values of λ_0 with $\zeta = 0.4$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	42.26	42.56	43.06	43.79	44.76	45.97	47.42	49.06	50.88
0.100	44.05	44.37	44.90	45.65	46.65	47.90	49.37	51.06	52.90
0.150	45.26	45.57	46.08	46.80	47.77	48.98	50.41	52.04	53.81
0.200	46.19	46.48	46.95	47.62	48.53	49.67	51.03	52.57	54.25
0.250	46.96	47.22	47.63	48.25	49.09	50.15	51.42	52.86	54.43
0.300	47.61	47.83	48.19	48.74	49.50	50.48	51.66	53.01	54.46
0.350	48.18	48.35	48.65	49.14	49.83	50.72	51.82	53.06	54.41
0.400	48.68	48.81	49.05	49.47	50.08	50.90	51.90	53.05	54.29
0.500	49.54	49.56	49.69	49.97	50.44	51.10	51.94	52.92	53.98
0.600	50.24	50.16	50.17	50.33	50.67	51.19	51.88	52.71	53.61
0.700	50.83	50.65	50.55	50.60	50.81	51.21	51.77	52.46	53.22
0.800	51.33	51.05	50.86	50.79	50.90	51.18	51.62	52.20	52.84
0.900	51.75	51.39	51.10	50.94	50.94	51.12	51.46	51.93	52.47
0.999	52.12	51.67	51.29	51.04	50.95	51.04	51.29	51.67	52.12

The effect of a temperature-dependent viscosity on the dimensionless pressure drop, in the case of forced convection, is illustrated qualitatively in Fig. 2, where plots of λ_0 versus γ are reported. Plot (a) refers to a fluid with constant viscosity; plot (b) refers to the case $\xi = -2$ and $\zeta = 0.2$; plot (c) refers to the case $\xi = 2$ and $\zeta = 0.2$. Plot (b) shows that, if the inner wall is heated ($\xi < 0$), the variable viscosity reduces the viscous pressure drop for low values of γ and increases it for high values of γ. Plot (c) illustrates the case of outer wall heated ($\xi > 0$). It shows that, in this case, the temperature-dependent viscosity enhances the viscous pressure drop for every value of γ; for low values of γ, the effect of the variable viscosity is stronger than in the case of inner heating. The figure shows also that, in the limit of parallel plane channel, $\gamma \to 1$, the viscous pressure drop is independent of the heat flow direction.

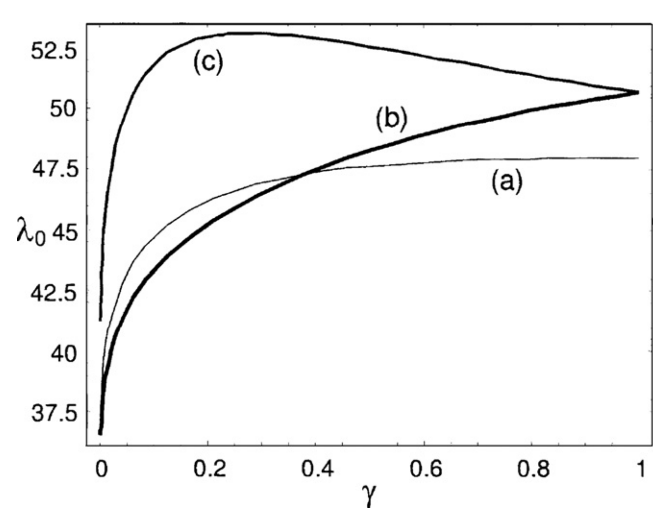

Fig. 2. Plots of λ_0 versus γ for a fluid with constant viscosity (a), $\xi = -2$ and $\zeta = 0.2$ (b), $\xi = 2$ and $\zeta = 0.2$ (c).

By employing Eqs. (29) and (33), the values of λ_1 as a function of γ and ξ have been calculated for $\zeta = 0.0$, $\zeta = 0.2$ and $\zeta = 0.4$. The results are reported in Tables 4–6, respectively. These tables show that λ_1 is an increasing function of ξ for every pair of values of γ and ζ. Moreover, λ_1 is negative for any value of ζ if $\xi \leqslant 0$ or if $\gamma \leqslant 0.10$. In

Table 4
Values of $\lambda_1 \times 10^2$ with $\zeta = 0.0$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	−7.197	−6.538	−5.882	−5.226	−4.569	−3.910	−3.251	−2.594	−1.943
0.100	−8.169	−7.243	−6.335	−5.439	−4.551	−3.669	−2.794	−1.928	−1.073
0.150	−8.659	−7.542	−6.452	−5.384	−4.330	−3.288	−2.257	−1.239	−0.2354
0.200	−8.908	−7.647	−6.419	−5.218	−4.037	−2.872	−1.720	−0.5827	0.5382
0.250	−9.011	−7.639	−6.305	−5.001	−3.719	−2.455	−1.206	0.02775	1.245
0.300	−9.019	−7.561	−6.144	−4.757	−3.395	−2.051	−0.7230	0.5904	1.887
0.350	−8.962	−7.438	−5.954	−4.502	−3.075	−1.666	−0.2727	1.107	2.471
0.400	−8.860	−7.285	−5.749	−4.245	−2.764	−1.302	0.1455	1.581	3.002
0.500	−8.572	−6.927	−5.318	−3.738	−2.180	−0.6383	0.8921	2.413	3.925
0.600	−8.222	−6.537	−4.885	−3.257	−1.649	−0.05403	1.533	3.115	4.693
0.700	−7.846	−6.142	−4.465	−2.810	−1.170	0.4597	2.086	3.710	5.335
0.800	−7.464	−5.754	−4.067	−2.397	−0.7393	0.9124	2.563	4.217	5.876
0.900	−7.087	−5.381	−3.692	−2.017	−0.3507	1.313	2.978	4.651	6.333
0.999	−6.726	−5.028	−3.344	−1.671	−0.00334	1.664	3.338	5.021	6.718

Table 5
Values of $\lambda_1 \times 10^2$ with $\zeta = 0.2$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	−7.087	−6.424	−5.764	−5.104	−4.444	−3.784	−3.125	−2.470	−1.822
0.100	−8.027	−7.098	−6.187	−5.288	−4.398	−3.516	−2.641	−1.778	−0.9276
0.150	−8.504	−7.385	−6.293	−5.221	−4.166	−3.123	−2.094	−1.078	−0.07980
0.200	−8.751	−7.487	−6.258	−5.054	−3.871	−2.705	−1.554	−0.4192	0.6967
0.250	−8.858	−7.484	−6.148	−4.841	−3.557	−2.291	−1.043	0.1889	1.401
0.300	−8.874	−7.415	−5.994	−4.605	−3.240	−1.894	−0.5658	0.7460	2.039
0.350	−8.829	−7.302	−5.816	−4.360	−2.930	−1.518	−0.1239	1.255	2.616
0.400	−8.739	−7.161	−5.622	−4.114	−2.630	−1.165	0.2844	1.719	3.138
0.500	−8.478	−6.830	−5.217	−3.632	−2.070	−0.5241	1.009	2.531	4.042
0.600	−8.154	−6.467	−4.810	−3.177	−1.564	0.03622	1.627	3.211	4.789
0.700	−7.803	−6.096	−4.415	−2.754	−1.109	0.5267	2.157	3.785	5.410
0.800	−7.444	−5.731	−4.039	−2.364	−0.7000	0.9574	2.613	4.271	5.931
0.900	−7.087	−5.378	−3.685	−2.004	−0.3320	1.337	3.008	4.685	6.369
0.999	−6.744	−5.044	−3.356	−1.677	−0.00316	1.671	3.349	5.037	6.737

Table 6
Values of $\lambda_1 \times 10^2$ with $\zeta = 0.4$

γ	ξ								
	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
0.050	−6.975	−6.308	−5.644	−4.981	−4.319	−3.658	−3.000	−2.347	−1.704
0.100	−7.884	−6.952	−6.038	−5.136	−4.245	−3.363	−2.490	−1.630	−0.7853
0.150	−8.348	−7.226	−6.132	−5.058	−4.001	−2.959	−1.931	−0.9196	0.07297
0.200	−8.592	−7.327	−6.095	−4.889	−3.705	−2.539	−1.389	−0.2579	0.8525
0.250	−8.704	−7.328	−5.989	−4.680	−3.394	−2.128	−0.8806	0.3481	1.555
0.300	−8.729	−7.267	−5.844	−4.452	−3.085	−1.738	−0.4097	0.8998	2.188
0.350	−8.694	−7.166	−5.676	−4.217	−2.784	−1.371	0.02391	1.401	2.758
0.400	−8.617	−7.037	−5.494	−3.983	−2.495	−1.028	0.4224	1.856	3.272
0.500	−8.382	−6.732	−5.115	−3.526	−1.959	−0.4102	1.125	2.648	4.157
0.600	−8.085	−6.395	−4.734	−3.096	−1.478	0.1263	1.720	3.306	4.883
0.700	−7.759	−6.049	−4.364	−2.698	−1.047	0.5934	2.228	3.858	5.484
0.800	−7.423	−5.707	−4.011	−2.330	−0.6606	1.002	2.662	4.323	5.985
0.900	−7.086	−5.375	−3.677	−1.991	−0.3132	1.362	3.038	4.718	6.404
0.999	−6.760	−5.058	−3.366	−1.682	−0.00298	1.676	3.360	5.052	6.754

the limit $\gamma \to 1$, the function $\lambda_1(\xi)$ is antisymmetric with respect to $\xi = 0$, as expected. For a fluid with constant viscosity, i.e., with $\xi = 0$ and $\zeta = 0, \lambda_1$ vanishes in the limit $\gamma \to 1$, in agreement with the results obtained in Ref. [8].

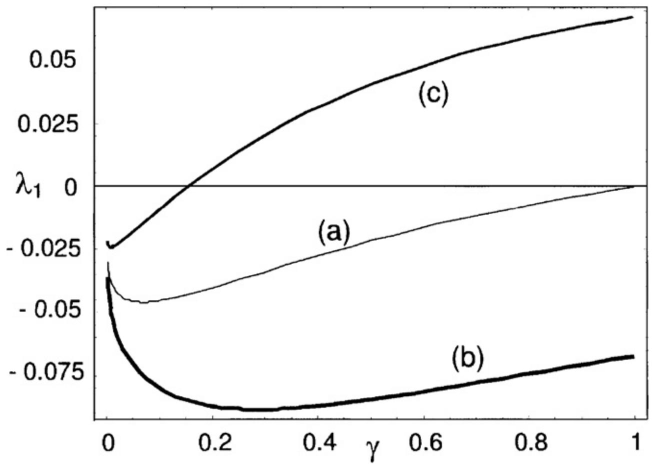

Fig. 3. Plots of λ_1 versus γ for a fluid with constant viscosity (a), $\xi = -2$ and $\zeta = 0.2$ (b), $\xi = 2$ and $\zeta = 0.2$ (c).

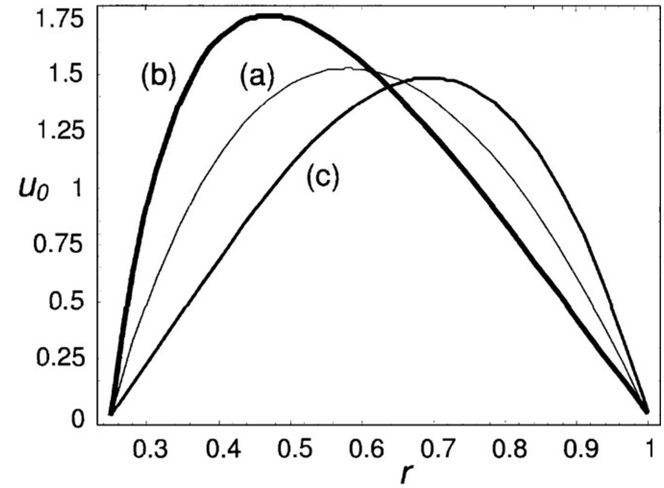

Fig. 4. Plots of u_0 versus r with $\gamma = 0.25$, for a fluid with constant viscosity (a), $\xi = -2$ and $\zeta = 0.2$ (b), $\xi = 2$ and $\zeta = 0.2$ (c).

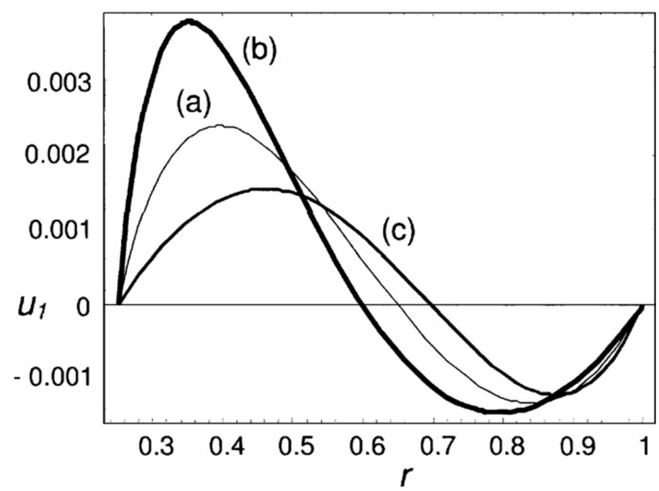

Fig. 5. Plots of u_1 versus r with $\gamma = 0.25$, for a fluid with constant viscosity (a), $\xi = -2$ and $\zeta = 0.2$ (b), $\xi = 2$ and $\zeta = 0.2$ (c).

This limit is not completely reached in Table 4, where the highest value of γ considered is 0.999.

The combined effect of buoyancy forces and of a temperature-dependent viscosity on the dimensionless pressure drop is illustrated qualitatively in Fig. 3, where plots of λ_1 versus γ are reported. Again, plot (a) refers to a fluid with constant viscosity; plot (b) refers to the case $\xi = -2$ and $\zeta = 0.2$; plot (c) refers to the case $\xi = 2$ and $\zeta = 0.2$.

Plot (a) shows that, for a fluid with constant viscosity, λ_1 is negative for every value of γ and tends to zero when $\gamma \to 1$. As a consequence, buoyancy forces reduce the viscous pressure drop when $Gr/Re > 0$ (inner heating and upward flow, or outer heating and downward flow) and enhance it when $Gr/Re < 0$. Plot (b) illustrates the effect of variable viscosity on λ_1 for a case of inner heating ($T_1 > T_2$, hence $\xi < 0$). It shows that, if the inner wall is heated, for any value of γ, λ_1 is negative and has an absolute value higher that in the case of constant viscosity. Therefore, for inner heating, the temperature-dependent viscosity reduces the viscous pressure drop when $Gr/Re > 0$ and enhances it when $Gr/Re < 0$. Clearly, if Gr/Re is positive and has a sufficiently high value, negative values of the dimensionless pressure drop λ may occur, i.e., the difference between the pressure and the hydrostatic pressure may increase in the flow direction. On the other hand, plot (c) shows that, for outer heating and variable viscosity, λ_1 is positive except for very low values of γ. Thus, if γ is not too small, buoyancy forces enhance the viscous pressure drop for upward flow and reduce it for downward flow, so that negative values of the dimensionless pressure drop λ may occur for downward flow.

The effect of a temperature-dependent viscosity on the velocity distribution, in the case of forced convection, is illustrated in Fig. 4, where plots of u_0 versus r are reported, for $\gamma = 0.25$. Plot (a) refers to a fluid with constant viscosity; plot (b) refers to the case $\xi = -2$ and $\zeta = 0.2$; plot (c) refers to the case $\xi = 2$ and $\zeta = 0.2$. Plots (b) and (c) show that the variable viscosity enhances the fluid velocity close to the heated wall and reduces it close to the cooled wall.

The combined effect of buoyancy and of a temperature-dependent viscosity on the velocity distribution is illustrated in Fig. 5, where plots of u_1 versus r are reported, for $\gamma = 0.25$. Again, plot (a) refers to a fluid with constant viscosity; plot (b) refers to the case $\xi = -2$ and $\zeta = 0.2$; plot (c) refers to the case $\xi = 2$ arid $\zeta = 0.2$. Let us discuss the interpretation of the figure with reference to upward flow. In this case, U_0 is positive and Gr/Re has the sign of $T_1 - T_2$. Plot (a) holds both for inner heating and for outer heating. It shows that for inner heating ($Gr/Re > 0$) buoyancy enhances the fluid velocity close to the inner wall and reduces it close to the outer wall. In the neighborhood of this wall, u_1 is negative and, if Gr/Re has a sufficiently high value, a negative velocity (flow reversal) may occur. For outer heating, Gr/Re is negative and a flow reversal may occur close to the inner wall. Plot (b) refers to inner heating. It shows that, in this case, the variable viscosity enhances the effect of buoyancy. Plot

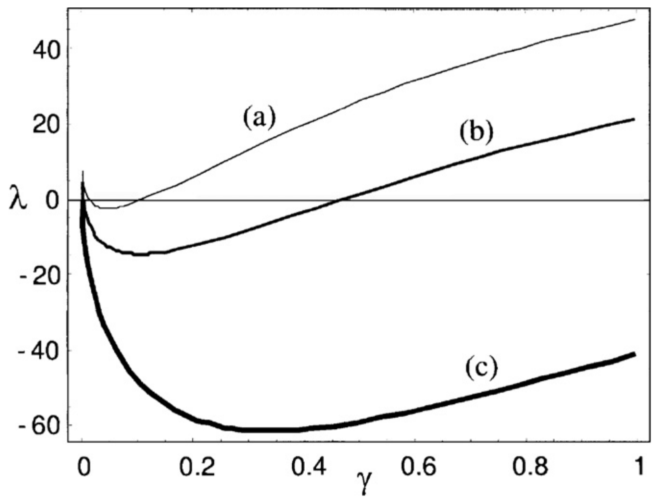

Fig. 6. Plots of λ versus γ for inner wall heated and $Gr/Re = 1000$, for a fluid with constant viscosity (a), water (b) and castor oil (c).

(c) shows that, for outer heating, the variable viscosity reduces the effect of buoyancy. The interpretation of Fig. 5 for downward flow, where Gr/Re and $T_1 - T_2$ have, opposite signs, is left to the reader.

Finally, in Figs. 6 and 7 the effects of a variable viscosity on the viscous pressure drop and on the velocity distribution are illustrated through specific examples, with reference to inner heating and upward flow with $Gr/Re = 1000$. Two fluids have been considered: water and castor oil. The latter is a high-viscosity liquid, employed in food industry. The dimensionless coefficients ξ and ζ have been determined through the viscosity values reported in Ref. [31], by assuming $T_1 = 60\,^{\circ}\mathrm{C}$ and $T_2 = 20\,^{\circ}\mathrm{C}$. The following values have been obtained: for water, $\xi = -0.8435$ and $\zeta = 0.2079$; for castor oil $\xi = -2.821$ and $\zeta = 0.6699$. Plots of λ versus γ are reported in Fig. 6: plot (a) refers to a fluid with constant viscosity, plot (b) refers to water and plot (c) refers to castor oil. The figure shows that, for inner heating and upward flow, the variable viscosity reduces for the viscous pressure drop. Indeed, with $Gr/Re = 1000$, negative values of λ occur for water if $0.004 \leqslant \gamma \leqslant 0.472$ and for castor oil for every value of γ. Plots of u versus r are reported in Fig. 7, for $Gr/Re = 1000$ and $\gamma = 0.25$, with reference to a fluid with constant viscosity (a), to water (b), and to castor oil (c). The figure shows that, in the conditions examined, the variable viscosity enhances the fluid velocity close to the inner wall and widens the flow reversal region close to the outer wall.

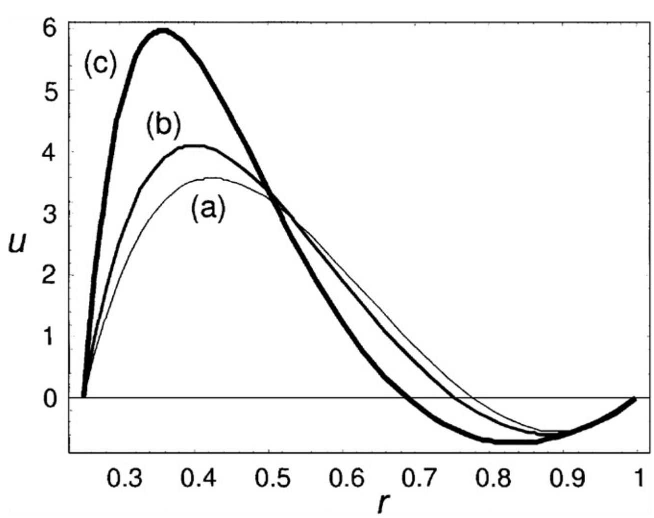

Fig. 7. Plots of u versus r for inner wall heated, $\gamma = 0.25$ and $Gr/Re = 1000$, for a fluid with constant viscosity (a), water (b) and castor oil (c).

5. Conclusions

The steady and laminar mixed convection with a temperature-dependent viscosity in a vertical annular duct with uniform wall temperatures has been studied analytically, by assuming that the flow is purely axial and that the fluid density is a linear function of temperature. Analytical expressions of the dimensionless velocity distribution, of the dimensionless pressure drop and of the Fanning friction factors have been provided. It has been pointed out that the dimensionless pressure drop is proportional to the cross-section-averaged Fanning friction factor only if the mean fluid temperature in any cross-section is chosen as the reference temperature in the linear equation of state of the fluid. The results show that the combined effects of buoyancy forces and of a variable viscosity on the dimensionless pressure drop and on the velocity distribution may be important, and that negative values of the viscous pressure drop may occur for inner wall heated and upward flow, as well as for outer wall heated and downward flow.

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